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# Linear multifractional multistable motion: LePage series representation and modulus of continuity

HERMINE BIERMÉ AND CÉLINE LACAUX

**Abstract** - In this paper, we obtain an upper bound of the modulus of continuity of linear multifractional multistable random motions. Such processes are generalizations of linear multifractional  $\alpha$ -stable motions for which the stability index  $\alpha$  is also allowed to vary in time. In the case of linear multifractional  $\alpha$ -stable motions, we improve the recent result of [2]. The main idea is to consider some conditionnally sub-Gaussian LePage series representations to fit the framework of [5].

**Key words and phrases** : stable and multistable random fields, modulus of continuity.

**Mathematics Subject Classification** (2010) : 60G17 60G22 60G52

## 1 Introduction

Self-similar random fields are required to model persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g. [1, 22]. The fractional Brownian motion ([15, 9]) provides the most famous self-similar model. Nevertheless, in image modeling, in finance or in biology for example, the phenomena under study are rarely Gaussian. Then,  $\alpha$ -stable random processes have been proposed as an alternative to Gaussian modeling, since they allow to model data with heavy tails, such as in internet traffic [16]. The linear fractional stable motion, which has been proposed in [21, 14], is one of the numerous stable extensions of the fractional Brownian motion. Let us recall how this self-similar random motion can be defined through a stochastic integral representation. To this way, let us consider  $H_1 \in (0, 1)$ ,  $\alpha_1 \in (0, 2)$  and  $M_{\alpha_1}$  a real-valued symmetric  $\alpha_1$ -stable random measure with Lebesgue control measure (see [17] p.281 for details on such measures). Then, a linear fractional stable motion is defined by

$$X_{\alpha_1, H_1}(t) = \int_{\mathbb{R}} f_+(\alpha_1, H_1, t, \xi) M_{\alpha_1}(d\xi), \quad t \in \mathbb{R} \quad (1.1)$$

where  $f_+$  is defined by

$$f_+(\alpha_1, H_1, t, \xi) = (t - \xi)_+^{H_1-1/\alpha_1} - (-\xi)_+^{H_1-1/\alpha_1} \quad (1.2)$$

with for  $c \in \mathbb{R}$ ,

$$(x)_+^c = \begin{cases} x^c & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Since the self-similarity property is a global property which can be too restrictive for applications, a multifractional generalization  $X_{\alpha_1, h}$  of this process has also been introduced by [18] to model internet traffic, by replacing  $H_1$  by a real function  $h$  with values on  $(0, 1)$ . Some necessary and sufficient conditions for the stochastic continuity of the linear multifractional stable motion  $X_{\alpha_1, h}$  have been given in [18] and its Hölder sample path regularity has been studied in [19]. The Hölder sample path properties have also been improved in [2] by establishing upper and lower bounds for the modulus of continuity. In the following, we will improve the upper bound, using the results we established in [5]. Let us mention that in the case where  $h \equiv H_1$  is constant, that is when  $X_{\alpha_1, h}$  is a linear fractional stable motion, sample path regularity properties have previously been studied in [17, 20, 10].

Moreover, the framework of [5] allows to study  $X_{\alpha_1, h}$  as well as some multistable generalizations for which the stability index  $\alpha_1$  is also allowed to vary with  $t$ . Multistable processes have been defined in [7] using sums over Poisson processes or in [6] using a Klass-Ferguson LePage series.

In this paper we consider a random field  $S_m$  defined using a LePage series representation of the linear fractional  $\alpha_1$ -stable motion and such that

$$S_m(\alpha(t), h(t), t), \quad t \in \mathbb{R}$$

is a linear multifractional multistable motion. This auxiliary random field  $S_m$  allows to study the variations due to the functions  $\alpha$ ,  $h$  and to the position  $t$  separately. Then, to study sample path regularity of linear multistable motions, our first step is to establish an upper bound for the modulus of continuity of the field  $S_m$  considering a conditionnally sub-Gaussian representation and applying [5]. The main property of sub-Gaussian random variables, which have been introduced by [8], is that their tail distributions decrease exponentially as the Gaussian ones. This property is one of the main tool used in [5] to study the sample path regularity property of conditionnally sub-Gaussian random series.

The paper is organized as follows. Section 2 introduces LePage series random fields under study. An upper bound of their modulus of continuity and a rate of convergence are stated in Section 3. Section 4 focuses on linear multifractional multistable motions. Some technical proofs are postponed to the appendix for reader convenience.

## 2 LePage series models

In order to define LePage series, let us introduce some notation.

**Hypothesis 2.1** Let  $(g_n)_{n \geq 1}$ ,  $(\xi_n)_{n \geq 1}$  and  $(T_n)_{n \geq 1}$  be three independent sequences of random variables satisfying the following conditions.

1.  $(g_n)_{n \geq 1}$  is a sequence of independent identically distributed (i.i.d.) real-valued symmetric sub-Gaussian random variables, that is such that there exists  $s \in [0, +\infty)$  for which

$$\forall \lambda \in \mathbb{R}, \quad \mathbb{E}(e^{\lambda g_n}) \leq e^{\frac{s^2 \lambda^2}{2}}. \quad (2.3)$$

2.  $(\xi_n)_{n \geq 1}$  is a sequence of i.i.d. random variables with common law

$$\mu(d\xi) = m(\xi)d\xi$$

equivalent to the Lebesgue measure (that is such that  $m(\xi) > 0$  for almost every  $\xi$ ).

3.  $T_n$  is the  $n$ th arrival time of a Poisson process with intensity 1.

Let us now introduce the random field  $(S_m(\alpha, H, t))_{(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}}$  we study in this paper.

**Proposition 2.1 (LePage series representation)** *Assume that Hypothesis 2.1 is fulfilled and let  $f_+$  be defined by (1.2). Then, for any  $(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$ , the sequence*

$$S_{m,N}(\alpha, H, t) = \sum_{n=1}^N T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad N \geq 1 \quad (2.4)$$

*converges almost surely and its limit is denoted by*

$$S_m(\alpha, H, t) := \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n. \quad (2.5)$$

**Proof.** Let  $(\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$ . Then, since Hypothesis 2.1 holds, the variables

$$W_n := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad n \geq 1,$$

are i.i.d., symmetric and such that

$$\mathbb{E}(|W_1|^\alpha) = \mathbb{E}(|g_1|^\alpha) \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha d\xi < +\infty,$$

since  $g_1$  and  $\xi_1$  are independent (see e.g. [17]). Therefore, by Theorem 5.1 of [13], the sequence

$$\left( \sum_{n=1}^N T_n^{-1/\alpha} W_n \right)_{N \geq 1}$$

converges almost surely as  $N \rightarrow +\infty$ , that is  $(S_{m,N}(\alpha, H, t))_{N \geq 1}$  converges almost surely.  $\square$

Let us conclude this section by some remarks.

**Remark 2.1** According to Proposition 5.1 of [5], the finite dimensional distributions of  $S_m$  do not depend on  $m$  as soon as Condition 2 of Hypothesis 2.1 holds. Moreover, when studying the sample path regularity of  $S_m$ , Proposition 5.1 of [5] allows us to change  $m$  by a more convenient function  $\tilde{m}$  if necessary.

**Remark 2.2** When  $\alpha = \alpha_1 \in (0, 2)$  is fixed,  $(S_m(\alpha_1, H, t))_{(H,t) \in (0,1) \times \mathbb{R}}$  is an  $\alpha_1$ -stable symmetric random field, which can also be represented as an integral under an  $\alpha_1$ -stable random measure  $M_{\alpha_1}$  with Lebesgue control measure. More precisely, for every  $\alpha_1 \in (0, 2)$ ,

$$(S_m(\alpha_1, H, t))_{(H,t) \in (0,1) \times \mathbb{R}} \stackrel{fdd}{=} d_{\alpha_1}(Y_{\alpha_1}(H, t))_{(H,t) \in (0,1) \times \mathbb{R}} \quad (2.6)$$

where  $\stackrel{fdd}{=}$  means equality of finite distributions and

$$Y_{\alpha_1}(H, t) := \int_{\mathbb{R}} f_+(\alpha_1, H, t, \xi) M_{\alpha_1}(d\xi), \quad (H, t) \in (0, 1) \times \mathbb{R}, \quad (2.7)$$

for  $M_{\alpha_1}$  a real-valued symmetric  $\alpha_1$ -stable random measure with Lebesgue control measure and

$$d_{\alpha_1} := \mathbb{E}(|g_1|^{\alpha_1})^{1/\alpha_1} \left( \int_0^{+\infty} \frac{\sin x}{x^{\alpha_1}} dx \right)^{1/\alpha_1}. \quad (2.8)$$

One can check Equation (2.6) following the proof of Proposition 5.1 of [5] or Proposition 4.2 of [4], which is a consequence of Lemma 4.1 of [11].

### 3 Sample path properties

Several papers [20, 10, 18, 19, 2] have already investigated sample path properties of the linear fractional stable motion  $X_{\alpha_1, H_1}$  defined by Equation (1.1) or of its multifractional generalization  $X_{\alpha_1, h}$  defined on  $\mathbb{R}$  by

$$X_{\alpha_1, h}(t) := Y_{\alpha_1}(h(t), t), \quad t \in \mathbb{R} \quad (3.9)$$

where  $\alpha_1 \in (0, 2)$ ,  $Y_{\alpha_1}$  is given by (2.7) and  $h$  is a function with values in  $(0, 1)$ . In the following, we improve the upper bound of the global modulus of continuity of  $X_{\alpha_1, h}$  stated in [2]. Our first step is to establish an upper bound for the global modulus of continuity of the field  $S_m$  defined by (2.5) on a compact set  $K$  of  $(0, 2) \times (0, 1) \times \mathbb{R}$ . To obtain our upper bound, we use

the results we established in [5] on conditionally sub-Gaussian random series.

Let us first recall (see [17] for example) that the  $\alpha_1$ -stable random process  $X_{\alpha_1, H_1} = (Y_{\alpha_1}(H_1, t))_{t \in \mathbb{R}}$  is unbounded almost surely on each compact set with non-empty interior when  $H_1 < 1/\alpha_1$ . A similar result holds for  $S_m$  as stated in the following proposition.

**Proposition 3.1** *Assume that  $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [a, b] \subset (0, 2) \times (0, 1) \times \mathbb{R}$  with  $0 < \alpha_1 \leq \alpha_2 < 2$ ,  $0 < H_1 \leq H_2 < 1$  and  $a < b$ .*

1. *If  $H_1 < 1/\alpha_1$ , then the random field  $S_m$  is almost surely unbounded on  $K$ .*
2. *If  $H_1 = 1/\alpha_1$ , then  $S_m$  does not have almost surely continuous sample paths on the compact set  $K$ .*

**Proof.** By Equation (2.6)

$$(S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}} \stackrel{fdd}{=} d_{\alpha_1}(X_{\alpha_1, H_1}(t))_{t \in \mathbb{R}}, \quad (3.10)$$

where  $d_{\alpha_1}$  is defined by Equation (2.8) and  $X_{\alpha_1, H_1}$  is the linear fractional stable motion given by (1.1).

Let us first assume that  $H_1 < 1/\alpha_1$ . Then, since  $a < b$ , by Corollary 10.2.4 of [17],  $(S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}}$  is unbounded almost surely on the compact set  $[a, b]$ . It follows that

$$\sup_{(\alpha, H, t) \in K} |S_m(\alpha, H, t)| = +\infty \text{ a.s.}$$

since  $\sup_{(\alpha, H, t) \in K} |S_m(\alpha, H, t)| \geq \sup_{t \in [a, b]} |S_m(\alpha_1, H_1, t)|$ .

Let us now assume that  $H_1 = 1/\alpha_1$  (which implies that  $\alpha_1 > 1$ ). Then,

$$X_{\alpha_1, H_1} = (M_{\alpha_1}([0, t])\mathbf{1}_{t>0} + M_{\alpha_1}((t, 0])\mathbf{1}_{t<0})_{t \in \mathbb{R}}$$

is a Lévy  $\alpha_1$ -stable motion and by Equation (3.10), so is the process  $(S_m(\alpha_1, H_1, t))_{t \in \mathbb{R}}$ . Since  $\alpha_1 < 2$ , the stable motion  $(S_m(\alpha_1, 1/\alpha_1, t))_{t \in \mathbb{R}}$  is not a Brownian motion and then does not have almost surely continuous sample paths (see Exercice 2.7 p.64 of [12] for instance). This concludes the proof.  $\square$

Therefore, it remains to study the sample paths on a compact set

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (0, 2) \times (0, 1) \times \mathbb{R}$$

such that  $H_1 > 1/\alpha_1$ , which implies that  $\alpha_1 \in (1, 2)$  and  $H_1 > 1/2$ .

The main result of this paper is the following theorem, which states an upper bound for the modulus of continuity of  $S_m$  on  $K$ , and for some  $m$  a rate of uniform convergence on  $K$  for the series  $S_{m,N}$  defined by (2.4).

**Theorem 3.1** *Assume that Hypothesis 2.1 is fulfilled. Let  $S_{m,N}$  and  $S_m$  be defined by (2.4) and (2.5) and let us consider the compact set*

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1, 2) \times (1/2, 1) \times \mathbb{R}$$

with  $A > 0$  and  $H_1 > 1/\alpha_1$ .

1. *As  $N \rightarrow +\infty$ , the series  $(S_{m,N})_{N \geq 1}$  converges uniformly on  $K$  to  $S_m$  and almost surely*

$$\sup_{\substack{x, x' \in K \\ x \neq x'}} \frac{|S_m(x) - S_m(x')|}{\tau(x - x') \sqrt{|\log(\tau(x - x'))| + 1}} < +\infty$$

with  $\tau(z) = |\alpha| + |H| + |t|^{H_1 - 1/\alpha_1}$  for  $z = (\alpha, H, t) \in \mathbb{R}^3$ .

2. *For  $\eta > 0$ , let us consider  $m = m_\eta$  defined by*

$$m_\eta(\xi) = c_\eta |\xi|^{-1} (1 + |\log(|\xi|)|)^{-1-\eta}, \quad (3.11)$$

with  $c_\eta > 0$  such that  $\int_{\mathbb{R}} m_\eta(\xi) d\xi = 1$ . Then, almost surely

$$\sup_{N \geq 1} N^\varepsilon \sup_{x \in K} |S_{m_\eta, N}(x) - S_{m_\eta}(x)| < +\infty$$

for any  $\varepsilon \in (0, 1/\alpha_2 - 1/2)$ .

**Proof.** For all  $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$  and all integer  $n \geq 1$ , we consider

$$V_{m,n}(x) := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha}, \quad (3.12)$$

so that

$$S_{m,N}(x) = \sum_{n=1}^N T_n^{-1/\alpha} V_{m,n}(x) g_n \quad \text{and} \quad S_m(x) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_{m,n}(x) g_n.$$

Let us also remark that for all  $x = (\alpha, H, t) \in (0, 2) \times (0, 1) \times \mathbb{R}$ ,

$$\mathbb{E}(|V_{m,n}(x)|^\alpha) = \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha d\xi < +\infty.$$

Note that if in Equation (2.3) the sub-Gaussian parameter  $s$  of  $g_n$  is less than 1, Equation (2.3) also holds for  $s = 1$ . Moreover, if  $s$  is greater than 1 we may write  $V_{m,n}(x) g_n = (s V_{m,n}(x)) g_n / s$  so that  $g_n / s$  is sub-Gaussian with parameter 1. Hence without loss of generality we may and will assume that  $s = 1$ . It follows that  $(g_n)_{n \geq 1}$ ,  $(T_n)_{n \geq 1}$  and  $(V_{m,n})_{n \geq 1}$  are three independent sequences that satisfy Assumption 4 in [5] on  $(0, 2) \times (0, 1) \times \mathbb{R}$ . Then, by Theorem 4.2 of [5], the result follows once we prove  $\mathbb{E}(|V_{m,1}(x_0)|^2) < +\infty$

for some  $x_0 \in K$  and Equation (15) of [5] for  $p = 1$ , namely (in our setting) if there exists  $r > 0$  such that

$$\mathbb{E} \left( \left[ \sup_{\substack{x, x' \in K \\ 0 < \|x - x'\| \leq r}} \frac{|V_{m,1}(x) - V_{m,1}(x')|}{\tau(x - x')} \right]^2 \right) < +\infty. \quad (3.13)$$

The following proposition, whose proof is postponed to the appendix, allows to find some  $m$  satisfying such conditions.

**Proposition 3.2** *There exists a finite deterministic constant  $c_{3,1}(K) > 0$  such that a.s. for all  $x, x' \in K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$ ,*

$$|V_{m,1}(x) - V_{m,1}(x')| \leq c_{3,1}(K) \tau(x - x') h_{m,K}(\xi_1),$$

with, for almost every  $\xi \in \mathbb{R}$ ,

$$\begin{aligned} h_{m,K}(\xi) &= \max \left( m(\xi)^{-1/\alpha_1}, m(\xi)^{-1/\alpha_2} \right) (1 + |\log m(\xi)|) \\ &\quad \times \left( \mathbf{1}_{|\xi| \leq e} + |\xi|^{-1+H_2-1/\alpha_2} \log |\xi| \mathbf{1}_{|\xi| > e} \right). \end{aligned} \quad (3.14)$$

Let us first consider  $m = m_\eta$  given by (3.11) for some  $\eta > 0$ . In view of Proposition 3.2, since  $V_{m_\eta,1}(\alpha, H, 0) = 0$  for all  $(\alpha, H, 0) \in K$ , up to use a finite covering of  $K$ , it is enough to prove that there exists  $r > 0$  with

$$\mathbb{E} \left( h_{m_\eta,K}(\xi_1)^2 \right) < +\infty, \quad (3.15)$$

for  $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$  with  $\alpha_2 - \alpha_1 \leq r$ . One has

$$\begin{aligned} \mathbb{E}(h_{m_\eta,K}(\xi_1)^2) &= \int_{\mathbb{R}} h_{m_\eta,K}(\xi)^2 m_\eta(\xi) d\xi \\ &= \int_{|\xi| \leq e} + \int_{|\xi| > e} := I_1 + I_2. \end{aligned}$$

On the one hand,

$$\begin{aligned} I_1 &= \int_{|\xi| \leq e} m_\eta(\xi) \max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2}) (1 + |\log(m_\eta(\xi))|)^2 d\xi \\ &\leq c_{3,2}(\eta, K) \int_{|\xi| \leq e} |\xi|^{-1+2/\alpha_2} (1 + |\log(|\xi|)|)^{(1+\eta)(2/\alpha_1-1)} (1 + |\log(m_\eta(\xi))|)^2 d\xi, \end{aligned}$$

with  $c_{3,2}(\eta, K)$  a positive finite constant. It follows that  $I_1 < +\infty$  since  $\alpha_2 > 0$ . On the other hand,

$$\begin{aligned} I_2 &= \int_{|\xi| > e} m_\eta(\xi) \max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2}) (1 + |\log(m_\eta(\xi))|)^2 |\xi|^{2(H_2-1/\alpha_2)-2} \log(|\xi|)^2 d\xi \\ &\leq c_{3,3}(\eta, K) \int_{|\xi| > e} |\xi|^{2(H_2+1/\alpha_1-1/\alpha_2)-3} \log(|\xi|)^{(1+\eta)(2/\alpha_1-1)+2} (1 + |\log(m_\eta(\xi))|)^2 d\xi, \end{aligned}$$



with  $c_{3,3}(\eta, K)$  a positive finite constant. Since  $\alpha_1 > 1$ , note that  $\alpha_2 - \alpha_1 < 1 - H_2$  implies that  $H_2 + 1/\alpha_1 - 1/\alpha_2 < H_2 + \alpha_2 - \alpha_1 < 1$  and thus  $I_2 < +\infty$ . Therefore choosing  $r \in (0, 1 - H_2)$ , Equation (3.15) and then (3.13) hold for  $m = m_\eta$ . By Theorem 4.2 of [5],  $(S_{m_\eta, N})_{N \geq 1}$  and  $S_{m_\eta}$  satisfy 1. and 2. of the theorem.

Since for almost every  $\xi \in \mathbb{R}$  the map  $(\alpha, H, t) \mapsto f_+(\alpha, H, t, \xi)$  is continuous on  $K$ , by Assertion 2. of Proposition 5.1 of [5],  $S_m$  satisfies Assertion 1. whatever  $m$  is.

□

**Remark 3.1** Assertion 2. in Theorem 3.1 holds for any  $m$  satisfying Equation (3.15) instead of  $m_\eta$ .

## 4 Linear multifractional multistable and stable motions

From now on let us consider  $\alpha : \mathbb{R} \mapsto (0, 2)$  and  $h : \mathbb{R} \mapsto (0, 1)$  two continuous functions. Under Hypothesis 2.1, by Proposition 2.1, we may consider the linear multifractional multistable motion defined on  $\mathbb{R}$  by

$$\tilde{S}_m(t) := S_m(\alpha(t), h(t), t), \quad (4.16)$$

with  $S_m$  given by (2.5).

### 4.1 Regularity and rate of convergence

We may also define  $\tilde{S}_{m, N}(t) := S_{m, N}(\alpha(t), h(t), t)$ , for all  $N \geq 1$ . The following theorem is a direct consequence of Theorem 3.1.

**Theorem 4.1** *Let us consider  $\alpha : \mathbb{R} \mapsto (0, 2)$  and  $h : \mathbb{R} \mapsto (0, 1)$  two continuous functions and two real numbers  $a < b$ . Then let us set*

$$\alpha_1 = \min_{t \in [a, b]} \alpha(t), \alpha_2 = \max_{t \in [a, b]} \alpha(t) \text{ and } H_1 = \min_{t \in [a, b]} h(t).$$

*Assume that  $H_1 > 1/\alpha_1$  and that  $\alpha$  and  $h$  are  $(H_1 - 1/\alpha_1)$ -Hölder continuous functions on  $[a, b]$ .*

1. *Then, as  $N \rightarrow +\infty$ , the series  $(\tilde{S}_{m, N})_{N \geq 1}$  converges uniformly on  $[a, b]$  to  $\tilde{S}_m$  and almost surely*

$$\sup_{\substack{t, t' \in [a, b] \\ t \neq t'}} \frac{|\tilde{S}_m(t) - \tilde{S}_m(t')|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'|| + 1}} < +\infty.$$

2. Moreover if  $m = m_\eta$  is defined by (3.11) with  $\eta > 0$ , then, almost surely

$$\sup_{N \geq 1} N^\varepsilon \sup_{t \in [a, b]} \left| \tilde{S}_{m_\eta, N}(t) - \tilde{S}_{m_\eta}(t') \right| < +\infty$$

for any  $\varepsilon \in (0, 1/\alpha_2 - 1/2)$ .

Note that one can use  $\tilde{S}_{m_\eta, N}$  to simulate  $\tilde{S}_{m_\eta}$ . The error of approximation is then given by  $N^\varepsilon$ .

## 4.2 Stochastic integral and series representation

Assuming that  $\alpha$  is a constant function equal to  $\alpha_1$ , we have already seen that  $\tilde{S}_m \stackrel{fdd}{=} d_{\alpha_1} X_{\alpha_1, h}$  where  $X_{\alpha_1, h}$  is the linear multifractional  $\alpha_1$ -stable motion defined by (3.9) and  $d_{\alpha_1}$  is given by (2.8). Using the previous theorem we will prove the following one.

**Theorem 4.2** *Let  $\alpha_1 \in (0, 2)$  and  $h : \mathbb{R} \mapsto (0, 1)$  be a continuous function. Let us also consider  $X_{\alpha_1, h}$  the linear multifractional  $\alpha_1$ -stable motion defined by (3.9) and two real numbers  $a < b$ . If  $H_1 := \min_{t \in [a, b]} h(t) > 1/\alpha_1$  and if  $h$  is  $(H_1 - 1/\alpha_1)$ -Hölder continuous on  $[a, b]$ , then there exists a continuous modification  $X_{\alpha_1, h}^*$  of  $X_{\alpha_1, h}$  such that almost surely*

$$\sup_{\substack{t, t' \in [a, b] \\ t \neq t'}} \frac{|X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t')|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||| + 1}} < +\infty.$$

**Proof.** Let  $\alpha : \mathbb{R} \rightarrow (0, 2)$  be the constant function equal to  $\alpha_1$  and let  $\tilde{S}_m$  be defined by (4.16). Since  $\tilde{S}_m \stackrel{fdd}{=} d_{\alpha_1} X_{\alpha_1, h}$  with  $d_{\alpha_1} \neq 0$  defined by (2.8), by Theorem 4.1, we already know that a.s.

$$\sup_{\substack{t, t' \in [a, b] \cap \mathcal{D} \\ t \neq t'}} \frac{|X_{\alpha_1, h}(t) - X_{\alpha_1, h}(t')|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||| + 1}} < +\infty,$$

where  $\mathcal{D}$  is the dense set of dyadic real numbers. Moreover, since  $h$  is continuous with values in  $(0, 1)$ , the stochastic continuity of the linear multifractional  $\alpha_1$ -stable motion  $X_{\alpha_1, h}$  has been established in [19]. This implies that there exists a modification  $X_{\alpha_1, h}^*$  of  $X_{\alpha_1, h}$  such that

$$\sup_{\substack{t, t' \in [a, b] \\ t \neq t'}} \frac{|X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t')|}{|t - t'|^{H_1 - 1/\alpha_1} \sqrt{|\log |t - t'||| + 1}} < +\infty,$$

see e.g. Section D.2 of [5] for the construction of  $X_{\alpha_1, h}^*$ . Then, the proof is complete.  $\square$

In [2], using a wavelet series expansion, under our assumptions of Proposition 3.9, the authors obtained a continuous modification  $X_{\alpha_1, h}^*$  satisfying a.s. for all  $\eta > 0$ ,

$$\sup_{\substack{t, t' \in [a, b] \\ t \neq t'}} \frac{|X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t')|}{|t - t'|^{H_1 - 1/\alpha_1} (|\log |t - t'|| + 1)^{2/\alpha_1 + \eta}} < +\infty.$$

Since  $1/2 < 2/\alpha_1$ , our result is sharper. Moreover it is quasi-optimal since, for  $\eta > 0$ , one can find  $h$  such that a.s.

$$\sup_{\substack{t, t' \in [a, b] \\ t \neq t'}} \frac{|X_{\alpha_1, h}^*(t) - X_{\alpha_1, h}^*(t')|}{|t - t'|^{H_1 - 1/\alpha_1} (|\log |t - t'|| + 1)^{-\eta}} = +\infty,$$

by Theorem 6.1 of [2]. Let us also quote that following our method based on [5], one may obtain an upper bound for the global modulus of continuity of linear fractional stable sheets, which is sharper than the one given in [3].

## A Proof of Proposition 3.2

Let us consider  $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1, 2) \times (1/2, 1) \times \mathbb{R}$  such that  $1/\alpha_1 < H_1 \leq H_2 < 1$ . Let us note that it is enough to prove Proposition 3.2 for  $A$  large enough. Then, in this proof, we assume, without loss of generality that  $A > e$  (so that  $\log \xi > 1$  for  $\xi > A$ ).

For all  $x = (\alpha, H, t) \in K$ , we set

$$\beta(x) = H - 1/\alpha \in (0, 1)$$

and remark that  $\beta(x) \in [\beta_1, \beta_2] \subset (0, 1)$  with

$$\beta_1 = H_1 - 1/\alpha_1 \text{ and } \beta_2 = H_2 - 1/\alpha_2.$$

Moreover, for all  $x = (\alpha, H, t) \in K$  and all  $\xi \in \mathbb{R}$ , let us note that

$$f_+(\alpha, H, t, \xi) = g(\beta(x), t, \xi)$$

with  $g$  defined on  $(0, 1) \times \mathbb{R} \times \mathbb{R}$  by

$$g(\beta, t, \xi) := (t - \xi)_+^\beta - (-\xi)_+^\beta.$$

Let us now consider  $x = (\alpha, H, t) \in K$  and  $x' = (\alpha', H', t') \in K$ . Then, by (3.12),

$$V_{m, n}(x) - V_{m, n}(x') = \left( g(\beta(x), t, \xi_n) m(\xi_n)^{-1/\alpha} - g(\beta(x'), t', \xi_n) m(\xi_n)^{-1/\alpha'} \right).$$

Proposition 3.2 follows from the following lemma, which proof is given at the end of this section.

**Lemma A.1** *Let  $0 < \beta_1 \leq \beta_2 < 1$  and  $A > e$ .*

1. *There exists a finite positive constant  $c_1(A, \beta_1, \beta_2)$  such that for all  $\beta, \beta' \in [\beta_1, \beta_2]$ , all  $t, t' \in [-A, A]$  and all  $\xi \in \mathbb{R}$ ,*

$$|g(\beta, t, \xi) - g(\beta', t', \xi)| \leq c_1(A, \beta_1, \beta_2) \left( |t - t'|^{\beta_1} + |\beta - \beta'| \right) h_{A,1}(\xi, \beta_2)$$

*with*

$$h_{A,1}(\xi, c) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{c-1} \log |\xi| \mathbf{1}_{|\xi| > 2A}.$$

2. *Moreover, there exists a finite positive constant  $c_2(A, \beta_1)$  such that for all  $\beta \in [\beta_1, \beta_2]$  and  $t \in [-A, A]$ ,*

$$|g(\beta, t, \xi)| \leq c_2(A, \beta_1) h_{A,2}(\xi, \beta_2)$$

*with*

$$h_{A,2}(\xi, c) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{c-1} \mathbf{1}_{|\xi| > 2A}.$$

Setting for almost every  $\xi \in \mathbb{R}$

$$\begin{cases} F_1(x, x', \xi) &:= |g(\beta(x), t, \xi) - g(\beta(x'), t', \xi)| m(\xi)^{-1/\alpha}, \\ F_2(x, x', \xi) &:= |g(\beta(x'), t', \xi)| \left| m(\xi)^{-1/\alpha} - m(\xi)^{-1/\alpha'} \right|, \end{cases}$$

we then have

$$|V_{m,1}(x) - V_{m,1}(x')| \leq F_1(x, x', \xi_1) + F_2(x, x', \xi_1).$$

Before we apply Lemma A.1 to bound  $F_1$  and  $F_2$ , let us remark that for all  $\xi \in \mathbb{R}$ ,

$$h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2) \leq c_3(A, \beta_2) \left( \mathbf{1}_{|\xi| \leq e} + |\xi|^{\beta_2-1} \log |\xi| \mathbf{1}_{|\xi| > e} \right) \quad (\text{A.17})$$

with  $c_3(A, \beta_2)$  a finite positive constant, which does not depend on  $\xi$ . Then, combining this remark with Lemma A.1, for almost every  $\xi \in \mathbb{R}$ ,

$$F_1(x, x', \xi) \leq c_1(A, \beta_1, \beta_2) c_3(A, \beta_2) \left( |t - t'|^{\beta_1} + |\beta(x) - \beta(x')| \right) h_{m,K}(\xi)$$

with  $h_{m,K}$  defined by Equation (3.14). Since  $\alpha_1 > 1$ , by definition of the function  $\beta$ , it follows that for almost every  $\xi \in \mathbb{R}$ ,

$$F_1(x, x', \xi) \leq c_1(A, \beta_1, \beta_2) c_3(A, \beta_2) \tau(x - x') h_{m,K}(\xi),$$

with  $\tau(x - x') = |t - t'|^{\beta_1} + |H - H'| + |\alpha - \alpha'|$ .

Moreover, applying Assertion 2 of Lemma A.1, Equation (A.17) and the mean value theorem, for almost every  $\xi \in \mathbb{R}$ ,

$$F_2(x, x', \xi) \leq c_2(A, \beta_1) c_3(A, \beta_2) |\alpha - \alpha'| h_{m,K}(\xi).$$

In view of the previous computations, we have: almost surely,

$$|V_{m,1}(x) - V_{m,1}(x')| \leq c_{3,1}(K)\tau(x - x')h_{m,K}(\xi_1)$$

with  $c_{3,1}(K) := c_3(A, \beta_2)(c_1(A, \beta_1, \beta_2) + c_2(A, \beta_1))$ . This concludes the proof of Proposition 3.2.  $\square$

We conclude this section by the proof of Lemma A.1.

**Proof.** [Proof of Lemma A.1] Let  $0 < \beta_1 < \beta_2 < 1$  and  $A > e$ . Let  $\beta, \beta' \in [\beta_1, \beta_2] \subset (0, 1)$  and  $t, t' \in [-A, A]$ . Let us write for all  $\xi \in \mathbb{R}$ ,

$$|g(\beta, t, \xi) - g(\beta', t', \xi)| \leq g_1(\beta', t, t', \xi) + g_2(\beta, \beta', t, \xi)$$

with

$$\begin{cases} g_1(\beta', t, t', \xi) &:= |g(\beta', t', \xi) - g(\beta', t, \xi)| \\ g_2(\beta, \beta', t, \xi) &:= |g(\beta', t, \xi) - g(\beta, t, \xi)|. \end{cases}$$

**Step 1: Control of  $g_1$ .** Let us note that if  $t = t'$ ,  $g_1(\beta', t, t', \xi) = 0$  for all  $\xi \in \mathbb{R}$ . Then, in this step, we assume now, without loss of generality that  $t < t'$ . This implies that

$$g_1(\beta', t, t', \xi) = \begin{cases} 0 & \text{if } \xi \geq t' \\ (t' - \xi)^{\beta'} & \text{if } t \leq \xi < t' \\ |(t - \xi)^{\beta'} - (t' - \xi)^{\beta'}| & \text{if } \xi < t. \end{cases}$$

Let  $\xi \in \mathbb{R}$  with  $|\xi| > 2A$ . If  $\xi < 0$  it follows that  $\xi < t < t'$ . Since  $\beta' > 0$ , applying the mean value theorem,

$$g_1(\beta', t, t', \xi) \leq \beta' |t - t'| |c_{\xi, t, t'} - \xi|^{\beta' - 1}$$

with  $c_{\xi, t, t'} \in (t, t') \subset [-A, A]$ . Moreover, since  $|\xi| > 2A$

$$|c_{\xi, t, t'} - \xi| \geq |\xi| - |c_{\xi, t, t'}| \geq |\xi| - A \geq |\xi|/2$$

and then

$$g_1(\beta', t, t', \xi) \leq 2^{1-\beta'} |t - t'| |\xi|^{\beta' - 1}$$

since  $\beta' \in (0, 1)$ . Therefore, for  $|\xi| > 2A$ ,

$$g_1(\beta', t, t', \xi) \leq 4A |t - t'|^{\beta_1} |\xi|^{\beta_2 - 1} \quad (\text{A.18})$$

since  $|t - t'| \leq 2A$ ,  $\beta' \in [\beta_1, \beta_2] \subset (0, 1)$  and  $2A > 1$ .

Now let  $\xi \in \mathbb{R}$  with  $|\xi| \leq 2A$ . Since  $0 < \beta' < 1$ , we have

$$|a^{\beta'} - b^{\beta'}| \leq |a - b|^{\beta'}$$

for all  $a, b \geq 0$ . By definition of  $g$ , it follows that

$$g_1(\beta', t, t', \xi) \leq \left| (t' - \xi)_+ - (t - \xi)_+ \right|^{\beta'} \leq |t' - t|^{\beta'} \leq 2A|t' - t|^{\beta_1}$$

since  $-A \leq t < t' \leq A$ ,  $0 < \beta_1 \leq \beta' < 1$  and  $A > 1$ . From this last inequality and Equation (A.18), we deduce that for all  $\xi \in \mathbb{R}$ ,

$$g_1(\beta', t, t', \xi) \leq 4A|t - t'|^{\beta_1} h_{A,2}(\xi, \beta_2) \quad (\text{A.19})$$

with  $h_{A,2}(\xi, \beta_2) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{\beta_2-1} \mathbf{1}_{|\xi| > 2A}$ .

**Step 2: Control of  $g_2$ .** Let us recall that for all  $\xi \in \mathbb{R}$ ,

$$g_2(\beta, \beta', t, \xi) = \left| (t - \xi)_+^{\beta'} - (t - \xi)_+^\beta + (-\xi)_+^\beta - (-\xi)_+^{\beta'} \right|.$$

Then, applying the mean value theorem, for all  $\xi \in \mathbb{R}$ ,

$$g_2(\beta, \beta', t, \xi) \leq |\beta - \beta'| \sup_{\beta_1 \leq c \leq \beta_2} \left| (t - \xi)_+^c \log(t - \xi)_+ - (-\xi)_+^c \log(-\xi)_+ \right|$$

where for  $c > 0$ ,

$$(x)_+^c \log(x)_+ = \begin{cases} x^c \log x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let us first consider  $\xi \in [-2A, 2A]$ . Then,  $(-\xi)_+ \in [0, 2A]$  and  $(t - \xi)_+ \in [0, 3A]$  since  $t \in [-A, A]$ . Therefore,

$$g_2(\beta, \beta', t, \xi) \leq \tilde{c}_1(A, \beta_1, \beta_2) |\beta - \beta'| \quad (\text{A.20})$$

with

$$\tilde{c}_1(A, \beta_1, \beta_2) = 2 \max_{\beta_1 \leq c \leq \beta_2} \max_{0 < u \leq 3A} u^c |\log u| = 2 \max \left( \frac{1}{e\beta_1}, (3A)^{\beta_2} \log(3A) \right) < +\infty.$$

Let us now assume that  $\xi < -2A$ . Then,  $\xi < t$  and

$$g_2(\beta, \beta', t, \xi) \leq |\beta - \beta'| \sup_{\beta_1 \leq c \leq \beta_2} \left| (t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi) \right|$$

with  $t - \xi > 0$  and  $-\xi > 0$ . Let us remark that  $-\xi \in (-\xi/2, -3\xi/2)$  since  $-\xi > 0$  and that

$$-\xi/2 < -A - \xi \leq t - \xi \leq A - \xi < -3\xi/2$$

since  $t \in [-A, A]$  and  $\xi < -2A$ . Then, for each  $c \in [\beta_1, \beta_2] \subset (0, 1)$ , by the mean value theorem,

$$\left| (t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi) \right| \leq |u_{t,\xi,c}|^{c-1} (c |\log u_{t,\xi,c}| + 1)$$

with  $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$ . Since  $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$  and  $-\xi/2 > A > e$ , we get

$$|(t - \xi)^c \log(t - \xi) - (-\xi)^c \log(-\xi)| \leq 4|\xi|^{\beta_2-1} \log |\xi|$$

for all  $c \in [\beta_1, \beta_2] \subset (0, 1)$ . Hence, for  $\xi < -2A$ ,

$$g_2(\beta, \beta', t, \xi) \leq 4|\beta - \beta'| |\xi|^{\beta_2-1} \log |\xi|.$$

Note that this last inequality still holds for  $\xi > 2A$  since in this case,  $g_2(\beta, \beta', t, \xi) = 0$ .

Then, we have proved that for all  $\xi \in \mathbb{R}$ ,

$$g_2(\beta, \beta', t, \xi) \leq \tilde{c}_2(A, \beta_1, \beta_2) |\beta - \beta'| h_{A,1}(\xi, \beta_2) \quad (\text{A.21})$$

with  $\tilde{c}_2(A, \beta_1, \beta_2) = \max(\tilde{c}_1(A, \beta_1, \beta_2), 4)$  and

$$h_{A,1}(\xi, \beta_2) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{\beta_2-1} \log |\xi| \mathbf{1}_{|\xi| > 2A}.$$

**Step 3: Proof of Assertion 1.** It follows from Equations (A.19) and (A.21) choosing  $c_1(A, \beta_1, \beta_2) = \tilde{c}_2(A, \beta_1, \beta_2) + 4A \in (0, +\infty)$  and using the fact that  $h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2)$  since  $A > e$ .

**Step 4: Proof of Assertion 2.** Let us remark that

$$g(\beta', t', \xi) = g(\beta', t', \xi) - g(\beta', 0, \xi)$$

since  $g(\beta', 0, \xi) = (-\xi)_+^{\beta'} - (-\xi)_+^{\beta'} = 0$ . Hence, applying Equation (A.19) with  $t = 0$  and  $\beta' = \beta$ ,

$$|g(\beta', t', \xi)| \leq 4A |t'|^{\beta_1} h_{A,2}(\xi, \beta_2) \leq 4A^{\beta_1+1} h_{A,2}(\xi, \beta_2),$$

which concludes the proof.  $\square$

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